Democracy & Education

# Democratic Practices as Part of Mathematical Modeling in Schools

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**Abstract**

We focus on how democratic practices occur during the modeling activity "The Candy Bag of Dreams." With democratic practices, we refer to approaches supporting inclusivity and active participation aimed at empowering students. We investigated one group of preservice teachers implementing an optimizing modeling activity during practicum in two fifth-grade classes, where students modeled their dream candy bags. Students engaged in mathematical modeling where mathematics was embedded in their social context, and they negotiated meanings and developed perspectives. Through this, they experienced lived democracy by actively engaging in and through democratic practices.

We found democratic practices such as students inviting peers to negotiate, showing consideration for peers, respecting different opinions, standing up and arguing for own interests, constraining self-interest, and discussing maximum fairness. They exercised their rights and responsibilities as learners and, by doing so, became empowered. The preservice teachers provided room for students' dialogues, discussions, and spaces for disagreement, a mathematics classroom *for* and *as* democracy. Implications of this study suggest that mathematical modeling activities can offer fruitful grounds for empowering students through their lived democracy. However, careful considerations should be made to ensure students' democratic practices during their group work.

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## Introduction

Schools have an important role in developing students as democratic citizens. In mathematics education, students develop quantitative competencies essential for themselves and a democratic society. Mathematical competencies can help students identify how mathematics is applied in decision-making on critical societal issues. They can also enable them to reflect on the implications of these decisions. The mathematics classroom is not only a place to provide students with critical mathematical skills but also a place to facilitate students' democratic values and attitudes (Aguilar & Zavaleta, 2012). Vithal (1999) emphasized that students can experience democratic life in a mathematics classroom as they learn to listen to others' ideas, argue, make decisions, and critically analyze arguments made by authorities.

One potential way students can engage in *democratic practices* is through mathematical modeling. Democratic practices refer to approaches supporting inclusivity and active participation aimed at empowering students. It can be that students are listened to (by teachers and peers), meet peers with respect, practice their freedom of speech, exercise disagreement, and have opportunities to act and influence through argumentation (Hauge et al., 2022).

### *Mathematical Modeling and Democratic Practices*

Mathematical models are essential in describing, predicting, and communicating critical issues important for democracy (Barwell, 2018). These issues can significantly impact citizens' rights, freedom, and ability to manage their daily lives. However, the mathematics, the people behind the mathematics, and the decision-makers using the mathematical models are not perfect. Therefore, it is crucial that citizens have competencies in critiquing and reflecting on mathematics and its use in society (Skovsmose, 2023).

Mathematical modeling can be challenging for teachers and students (Niss & Blum, 2020), and including democratic perspectives can add to these challenges. For instance, focusing on democratic practices can lead to losing sight of the mathematics or vice versa. However, Blomhøj (2001) has argued that mathematical modeling can contribute to the general formation of students as participatory members of society. He refers to three broad goals: the ability to achieve self-determination, participate in decision-making, and demonstrate solidarity with citizens less privileged. In modeling, students must negotiate what assumptions to make when solving real-world problems, how to simplify reality into manageable mathematical entities, and what approaches to use.

### *Mathematical Modeling and Dialogical Interactions*

When modeling, group work is particularly suitable (Niss & Blum, 2020). However, this requires collaboration and dialogical interaction among group members. Dialogical interactions involve students raising questions regarding the problem, to peers and teachers, arguing about their perspectives, and commenting on peers' ideas. Artigue and Blomhøj (2013) emphasized students' dialogical interactions as crucial for developing citizenship and strengthening democracy and highlighted inquiry-based education as one way to facilitate students' learning.

Inquiry-based mathematics education refers to student-centered learning, where students collaborate, investigate, experiment, systematically work with variables, calculate, ask questions, discuss, critically reflect, and interact with peers and teachers in ways that contribute to the negotiation of meaning (Artigue & Blomhøj, 2013). Maaß et al. (2019) found that inquiry-based teaching approaches and socio-scientific issues, when modeling, could promote active citizenship. Dialogical and inquiry-based approaches during modeling activities can encourage student interaction and collaboration and provide an environment for them to practice citizenship.

### *Research Focus*

Combining mathematical modeling with democracy and citizenship is not a straightforward task. However, in a society where students increasingly encounter mathematical models, students need to experience negotiations and the dialogical inquiries involved in modeling processes, as well as have a critical attitude to the use of mathematical modeling and question the foundation of the existing models (Antonius, 2004).

Mathematical modeling has been included in mathematics curricula worldwide (Geiger et al., 2022). In Norway, it became a core element in the new curriculum (Ministry of Education and Research, 2017, 2019) and connected with the interdisciplinary topic "Democracy and Citizenship." It emphasizes that students learn to think critically, deal with conflicts of opinion, and respect disagreement. Further, students should develop democratic competencies in formulating their arguments and be conscious of mathematical models' underlying conditions and premises. Implementing modeling, democracy, and citizenship in the curriculum requires a closer look at how teachers can put these aims into practice in their teaching and learning.

This research focuses on getting insight into how democratic practices can be involved in constructing or applying a mathematical model in school. Our research question is: How do democratic practices occur during the modeling activity "The Candy Bag of Dreams"? We use empirical data from one group of preservice teachers (PTs) modeling activity in two fifth-grade classes.

### *Theoretical Frameworks in Mathematical Modeling*

Research within modeling often includes theoretical frameworks where the modeling process and modeling competencies are essential (Cevikbas et al., 2022; Niss & Blum, 2020). The *modeling process* is often visualized as a cyclic process where students start with a real situation, simplify this to a mathematical problem and model, and produce results addressing the initial problem (Niss & Blum, 2020). The various steps in the process are then used as a starting point to research how mathematics can be taught and learned.

In their systematic literature review, Cevikbas et al. (2022) defined *holistic modeling competence* as when students can perform

and reflect on the whole modeling process. They described four sub-competencies: action, meta-competence, critical competence, and social competence. The last two sub-competencies are relevant to this research. Critical competence is when students develop insight into relations between mathematics and reality and consider subjectivity; social competence is when students can work in groups and communicate about and via mathematics. Although modeling competencies and modeling processes are essential concepts and frameworks within mathematical modeling, democratic practices are not explicitly part of these constructs.

Previous research focusing on mathematical modeling and democratic practices include matters of social justice, and theoretical frameworks reflect these perspectives. Jung and Wickstrom (2023) described a conceptual framework connecting the modeling process components to social-justice-oriented issues. For instance, when defining the problem, this could include both a mathematical problem and a social justice problem; when defining variables, these should be based on concerns from both social justice and mathematical aspects.

Inspired by frameworks from culturally responsive teaching, Turner et al. (2024) identified three main strands: knowledge and identities, rigor and support, and power and participation, focusing on resisting marginalization during mathematical modeling. For instance, the last strands concern how intellectual authority can be distributed among teachers and students and how patterns can be disrupted and shape social interactions. The frameworks from Jung and Wickstrom (2023) and Turner et al. (2024) provide valuable insight into how theoretical perspectives can support research on social justice issues. However, our context does not explicitly focus on marginalized students but on the democratic practices occurring during the modeling activity.

By going through earlier research involving democratic practices and mathematics education and modeling, we identify three relevant aspects (see Figure 1) involving democratic practices that can be applied when modeling: Empowerment and mathemacy (to be able to understand society in term of mathematics), dialogues and discussions, and rights and responsibilities. In the following section, we elaborate on these aspects.

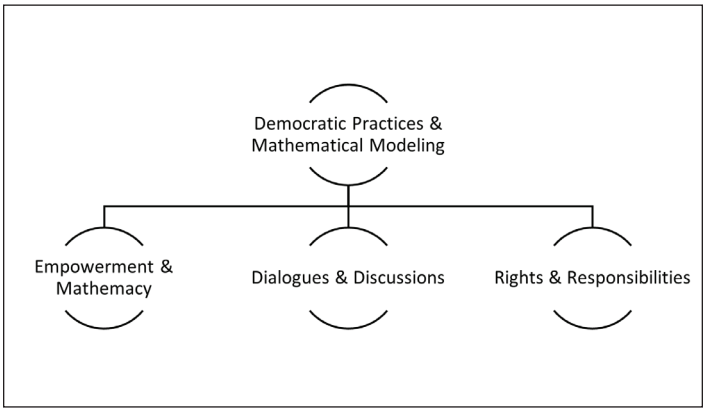


**Figure 1.** Democratic Practices and Mathematical Modeling

## Theoretical Perspective of Mathematical Modeling and Democratic Practices

Mathematical modeling is the process of solving real-world problems with mathematical concepts, tools, and techniques (Niss & Blum, 2020). When dealing with real-world issues and modeling in the classroom, aspects of democratic practices can come into play in multiple ways. The problem itself can involve democratic issues. It can affect how students work in the classroom, collaborate, and communicate through dialogues and discussions. It can involve the rights and responsibilities of students, and it can involve their learning outcomes.

Mathematical modeling, where students engage in a realistic situation, formulate a problem, set up a model for the problem, and suggest a solution based on mathematics, includes a range of various types of problems (Maaß, 2010). The problems differ in authenticity, context, relevance for students' lives, cognitive demand, and mathematical domain. One type of modeling involves optimizing problems, where students should find the best possible solution among a set of options and consider variables and assumptions that are not predefined. These modeling problems are, according to Greefrath et al. (2022), easily accessible and motivational for students.

Finding optimal solutions through mathematical modeling can involve maximizing or minimizing some variables (e.g., cost minimization or striving for maximum fairness) to determine the optimal time, budget, resources, or logistics (Eley, 2013; Suh et al., 2021). However, the choice of perspective is not obvious, and a solution that works well for one (individual perspective) could result in a conflict if it worsens the situation for others. Giving more weight to some variables is a choice you make as a modeler; for instance, if the modeler includes variables such as economics while leaving out ecological variables. Therefore, when students engage in these modeling problems, they need to consider which variables they include, reflect on fairness, and the potential impact of the model.

### *Democratic Modeling Practices Can Empower Students*

The mathematics classroom can be considered as education *for* or *as* democracy (Jansen et al., 2021). Jansen et al. described a mathematics education *for* democracy as learning competencies and practices relevant to society (e.g., critical thinking, argumentation to defend a stance, and quantitative thinking). Mathematics education *as* democracy can, according to Jansen et al., engage students in collective and fair decision-making processes, ensuring that everyone has a say and can speak up to authorities and accommodating multiple ways of thinking. As we see in the previous section, during mathematical modeling, the students can develop competencies for democracy, and at the same time, they can participate in decisions about such as assumption-making; thus, the mathematics classroom works as democracy.

Mathematical modeling provides the platform for promoting empowerment and developing mathemacy. Students' empowerment and mathemacy are central to supporting students as citizens (Skovsmose, 2023). Empowerment is understood as the capacity to speak up for oneself and participate in political discussions, which is essential in democratic practices. Empowerment can also

include speaking up for others and constraining self-interest, which includes moral principles and concerns for others' well-being (Mansbridge, 1990). In group modeling activities, especially when the problem involves different interests, the students must coordinate and co-decide the assumptions to make, approaches to use, etc., which may empower them to face problems outside the classroom. *Mathemacy* is the competency to interpret and understand our social reality using mathematical competencies (Skovsmose, 2023). Mathematics partly formats this reality by shaping how knowledge is produced, disseminated, and understood. Thus, mathematics is a powerful social and cultural practice that influences how we make sense of the world. An essential part of mathemacy is reflective knowing (Skovsmose, 2023), which involves questioning, challenging, and critically examining how mathematics is used and its implications. In this sense, one can argue that mathemacy is one of the ultimate goals of modeling classroom activities.

### *Students' Dialogues and Discussions as Democratic Modeling Practices*

Students' dialogue and discussions are essential to their modeling process, and the classroom where this activity occurs should represent basic democratic values. Alrø and Skovsmose (2002) argued that if learning is to support citizenship, dialogue must play an important role. Dialogues and discussions can allow students to engage in deeper learning and meaning-making and contribute to new understandings (Steffensen et al., 2022). It can take the form of oral interactions but also include written texts or bodily gestures. When students communicate during modeling, it is often driven by the students rather than the teacher. Student-led communication can encourage more democratic talk practices with opportunities to engage in peers' ideas and allow "students to think mathematically in equitable classroom discussions, where students' inquiries are pursued and valued" (Hansen, 2021, p. 283).

The competencies to have dialogues and discussions are not straightforward and should not be taken for granted. Instead, it must be developed through dialogic teaching and learning approaches that address dialogue, discussions, and active engagement (for an elaboration on dialogic teaching and learning, see Alrø & Skovsmose, 2002). Alrø and Skovsmose (2002) defined dialogue as an inquiry process where participants make inquiries, explore perspectives, are willing to suspend pre-understandings, experience unpredictability, run risks, and maintain equality. Their Inquiry-Cooperation (IC) model describes eight essential dialogue elements: get in contact, locate, identify, advocate, think aloud, reformulate, challenge, and evaluate. Qualities such as students empowered to advocate for their standpoint, think aloud, and challenge peers' perspectives are essential to students' democratic practices.

Alrø and Johnsen-Høines (2012) highlighted inquiry-based dialogues as a potent way of pursuing democratic practices in the classroom. They defined such dialogues as conversations with certain qualities, including equality, inquiry, unpredictability, risk-taking, and an open, curious, and wondering attitude toward the subject, dialogue partners, oneself, and the interactions. Qualities of inquiry-based dialogues involve true inquiring questions, and students should experience relevance to the problem. Further, inquiry-based dialogues should invite all dialogue partners to join, reflect a wish to understand more, have an open and curious attitude, and engage in dialogical listening (displayed through verbal and nonverbal gestures, see, e.g., Sjöblom, 2022; Steffensen et al., 2022). The dialogues should not strive for one correct answer but be a place where students can learn to express their opinions and present argumentation, respond to each other constructively, and respectfully learn from each other.

When students investigate real-world problems through modeling activities, it is, according to Barbosa (2007), essential to engage them in discussions where they reflect. He highlighted negotiation spaces, where "none of the parties wishes to impose his/her perspective, but rather put it up for discussion" (p. 239). Such spaces could involve negotiations on the choice of model or mathematical approaches. Related to this, Frejd and Bergsten (2018) suggested that negotiations should be made explicit in the teaching practice of modeling. They describe that modeling involves human negotiations about the problem, variables, function, and implications of the model.

Negotiation space could also involve the choice of pedagogical practice during modeling. De Loiola Araújo and da Silva Campos (2015) suggested that negotiation spaces between students and teachers can be used to "meet the needs of the pedagogical practice of the teacher" (p. 284). In modeling, an example of such a pedagogical practice is teachers and students having a genuine discussion about which mathematical tools would be appropriate to use in a situation. Such choices can be essential when modeling; for instance, using calculators and spreadsheets can shift the emphasis from following mathematical operations toward focusing on using mathematical knowledge (Greefrath et al., 2011).

Democratic practices rely on pedagogical choices, and in the mathematics classroom, communication structures can be rigid (teachers ask questions, they know the answers, and students answer). Therefore, Skovsmose (2023) suggested "scene setting" as one way to diminish the impacts of these structured settings. To avoid such communication structure, teachers can contribute to setting the scene by engaging students in real-world modeling problems or framing the problem in realistic ways. Furthermore, the pedagogical choices also involve defining teachers' roles in a modeling process. Finding the balance between the students' independence and the teacher's guidance can be challenging. However, as highlighted by Turner et al. (2024), the modeling activities can help teachers step back and give intellectual authority to the students. By that, the activity expands the roles available for students (perhaps in contrast to their "normal" classroom), giving them the necessary space to inquire, discuss, and explore the problem.

### *Students Exercising Their Rights and Responsibilities Through Democratic Modeling Practices*

Democratic practices can include students' rights and responsibilities as learners. Prasad and Kalinec-Craig (2021) described how students' rights are associated with their responsibility when creating a democratic mathematics classroom. The "right to

be confused" is related to the responsibility "to persevere through that confusion to some state of resolution" (p. 2). Modeling problems are open-ended and can allow students to exercise their right to be confused. Through inquiry-based learning and investigations, students learn perseverance and reach some resolution. The right "to claim a mistake and revise your thinking" can be connected with the responsibility to "address the mistake by reasoning about your thinking and returning to work that seems finished" (p. 2). Mathematical modeling requires constantly adjusting and evaluating, and learning to deal with previous assumptions or mistakes is essential.

The right to "speak, listen, and be heard" is associated with the responsibility to "share your thinking" and ensure peers "the opportunities to speak, listen, and be heard" (Prasad and Kalinec-Craig, 2021, p. 2). This right can involve both peers and teachers throughout the modeling process. The responsibility to share one's ideas is crucial, for instance, at the beginning of the process when making assumptions or when making critical reflections about the model. Also, paying attention so that peers share their thinking can contribute to including multiple perspectives. The right to "write, do, and represent only what makes sense to you" can relate to the responsibility to "notice in what ways you are engaging in the ideas on an individual level" (Prasad and Kalinec-Craig, 2021, p. 2) and ensure your peers' engagement in the same way. Choosing the variables during modeling can be an individual process, and the right to represent what makes sense to you can be one way of learning that mathematical modeling only partly reflects the real world; instead, it displays some chosen attributes of reality.

There can be tension between the learners' right and responsibility to speak and the right to be silent, as described by Prasad and Kalinec-Craig (2021) and Tyson et al. (2022). Silence can reveal how power is situated among students. Historically, certain humans have been actively silenced, for instance, based on gender or race. If unaware of the dynamics of classroom talks, teachers can unconsciously uphold such and other power relations. Tyson et al. described that silence is sometimes interpreted as a deficit, showing a lack of knowledge, inattention, or uncertainty, but that silence could also be a cultural practice. Although learning happens through talk, they highlighted that learning is an internal reflection on what is heard, where learners contemplate meanings and struggle to understand contradictions, confusion, or new ideas. They emphasized that students' right to be silent is equivalent to the right to speak, listen, and be heard.

Teachers can be essential in supporting students in claiming and enacting their democratic rights. Jansen et al. (2021) investigated how teachers can enact their governing role in the classroom while supporting students' democratic rights (as described by Prasad & Kalinec-Craig, 2021). Jansen et al. highlighted that group work is a safe opportunity to challenge teachers' authority, share unfinished and in-progress ideas, and provide opportunities to understand multiple perspectives (such as focusing on alternative solutions, representations, explanations, or justifications). Another aspect of enacting teachers' governing role can be through what Wright (2020) described as a socially just pedagogy that requires teachers to cultivate relational authority rather than relying on positional authority. While teachers who exercise *relational authority* negotiate classroom rules with students and explain the reasons behind pedagogical choices, teachers who depend on *positional authority* instead appeal to their position of power in imposing rules (Wright, 2020).

Teachers can support students by listening to what students say and encourage students toward a commitment to listening. Listening can provide insights into what students think. Hintz et al. (2018) explored how teachers use pedagogical listening while their students exercise their rights to be confused, make mistakes, and say and write what makes sense (as described by Kalinec-Craig, 2017). Hintz et al. argued that to promote equitable mathematical discussions, teachers should listen for more than predefined rights and wrongs (evaluative listening). Instead, teachers should be "listening to and for students' struggles to find and articulate new ideas and understandings" (p. 3). They described a framework for pedagogical listening, bringing together five types of teacher listening: self-reflective, empathic, educative, supportive, and generative. The various forms of listening are interconnected and support fostering an environment where students can engage with their own and others' insights as part of democratic practices.

Rights and responsibilities as democratic practices can be about rights to be confused, claim mistakes, revise thinking, speak, be silent, listen to, be heard, and represent what makes sense to you. It can be about the connected responsibilities to persevere through confusion, address mistakes, share your thinking, allow peers to share their thinking, and engage in ideas. It can be about teachers facilitating spaces for supporting students through group work, pedagogical listening, and cultivating relational authority.

## The "Candy Bag of Dreams" in a Norwegian Classroom

This research is part of a larger design research study at Western Norway University of Applied Sciences (LATACME, 2023), where we, among others, researched how PTs learn and teach mathematical modeling. This case study uses data from one group of PTs and their lessons in practicum. They were PTs in their second year and working with Grades 1–7. As their mandatory assignment, the PTs were required to design and implement a modeling activity for students in their practicum and describe and reflect upon the modeling activity in a written submission.

The assignment was accompanied by examples of modeling activities from the classroom, such as "Bungee Jumping with Barbie" (Wæge, 2007) and "3 Act Math" (Wallace & Jensen, 2017), where interactions between students and teachers reflect a distribution of power in favor of students. PTs wrote lesson plans before their practicum and shared them with other students and teacher educators. They presented their findings in plenary after practicum, received feedback from peers and teacher educators, and handed in their mandatory tasks after these activities.

The PTs designed the activity "The Candy Bag of Dreams" for their two fifth-grade classes (9–10-year-olds). Typically, in Norway, children buy or receive candies on Saturdays, and many share these with siblings or friends. Thus, this context is a part of the students' culture. The PTs invited students into the activity using real candies instead of textbook tasks; hence, they set the scene in line with

Skovsmose's ideas (2023). This contributed to students experiencing the task as real and becoming engaged in negotiations about which candy to include.

The two lessons lasted 45 minutes each, and the students worked in groups of three to four (four groups in each class). The PTs started the modeling lesson by bringing a bag of candies of different types to set the scene for the task (Figure 2). The accompanying text on the Smart Board was: "You can choose from the list every kind of candy you want, but each group can only have a maximum of 300 grams."


**Figure 2.** Various Types of Candies and the Weight (Left); Whiteboard Overview of the Weighted Candies (Right)

When the groups had reached the sum of (exactly) 300 grams, the PTs presented another aspect of the task: "Share the candies in your group as fairly as possible" (Figure 3).


**Figure 3.** Share the Candy as Fairly as Possible on Everybody in Your Group

The PTs formulated the task and domain of inquiry. They selected the 17 candies and chose to weigh them in front of the class for hygiene reasons, and they wrote a list of candies with their weights on the whiteboard. Thus, we interpret that the PTs let the students engage in parts of the modeling process, as described by Niss & Blum, 2020, focusing on mathematizing and working mathematically, where students only had to deal with problems that required a limited range of modeling competencies rather than holistic competence, as described by Cevikbas et al. (2022). We consider the activity as an optimizing modeling problem, as defined by Eley (2013), Greefrath et al. (2022), and Suh et al. (2022).

Four researchers observed and recorded both lessons. The research study follows the ethical guidelines from The Norwegian National Committees for Research Ethics (2022). Students, with consent, were placed in the same groups, and four groups were video- and audio-recorded. For later references, we gave the students fictive names: Levi, Chris, and Emma for Group 1; Adam, Irene, and Noah for Group 2; Theo, Alex, and Odin for Group 3; and Johan, Glen, and Tim for Group 4. The PTs were named Peter and Anna.

We (the researchers) transcribed and thematically coded the recordings in NVivo with other empirical data (e.g., pictures of students' notebooks, the mandatory task). The unit of analysis was the interactions between students, and between students and PTs. We watched and discussed the recordings together and used inductive and deductive coding. Examples of deductive coding are searching for elements such as advocating, thinking aloud, reformulating, inquiry, and wondering attitude from the IC model and inquiry-based dialogues (Alrø & Skovsmose, 2002; Alrø & Johnsen-Høines, 2012). Examples of inductive coding are observing students' dialogues (e.g., oral, gestures, body language) about fair criteria and arguments on tools and approaches. The refinement of coding was a continuous and collaborative process, and codes were organized in tables and mind maps for overviews when analyzing and synthesizing the findings.

## Findings and Discussions

An overview of the findings is presented in Figure 4, which provides a mind map of identified democratic practices placed in four overarching themes: (a) Which voice should be heard? (b) Bringing in new perspectives and missed opportunities, (c) Which tools and approaches do we take? And (d) What are fair criteria for sharing? We structure this part according to the four identified themes.

In the following, we elaborate on some chosen examples. These were selected because we considered them interesting examples (and not because we considered them exemplary), which gave us opportunities to gain insight and reflect on how democratic practices can occur during modeling.

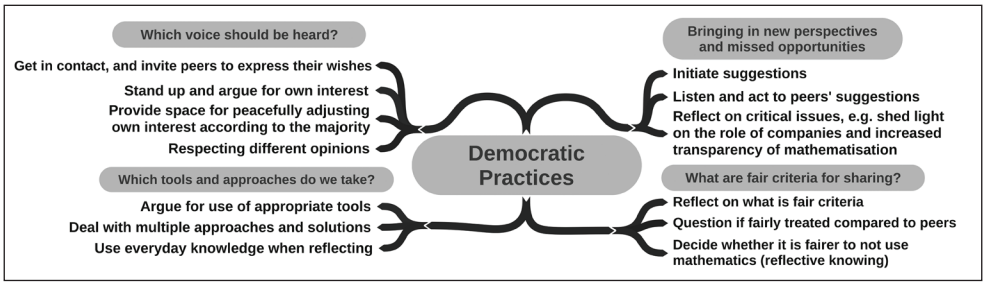


**Figure 4.** Overview of Identified Democratic Practices

### *Which Voice Should Be Heard?*

In most groups, students were inclusive and posed questions like "Does everybody want chocolate balls?" and waited for positive confirmation before continuing. In cases of negative responses, students typically commented something such as "Okay, what about . . ." These utterances from students' negotiations are attempts to get in contact with peers, and they try to identify and locate peers' opinions. When PT Anna asked how students decide which candy to include, one student said, "We vote on what to choose," and another supported this by saying, "So we all agree." Thus, an essential part of students' negotiation is including all group members' wishes and letting everyone have a say, which are key features in democratic practices. Students include peers' opinions by asking directly, writing down wishes, or voting. These traits of their dialogues, expressing and exercising concern for others and letting everyone have a say, can contribute to their capacity to speak for themselves and their empowerment. It involves the learners' rights to speak, listen, and be heard and their responsibility to share their thinking with peers. Letting everyone have a saying can include insight into modeling subjectivity; thus, one could argue that students enact critical competencies and show social modeling competencies.

Constraining self-interest by fairness to the majority is essential in democratic practices, and in Group 1, Chris first expressed a wish to have five strawberry candies. Emma immediately responded, "Fifty grams? Only you?" Then Chris said, "No, okay. Two, only two strawberries. Strawberries are the best." The comment made by Emma probably made Chris adjust his initial request. The dialogical interplay provides spaces for adjusting individual needs.

In contrast, in Group 4, three students passionately negotiated about including bamsemums (a big candy resembling a bear); Glen wanted to include this, while Johan did not. Their argumentation involved the taste, weight, number, and voting. After fierce negotiations, they concluded to have one bamsemums. They showed willingness and competencies to stand up and advocate for their interests, which are vital qualities in inquiry-based dialogues and empowerment. When PT Peter arrived, the following conversation took place:

> PT Peter: How is it going, boys?
> Glen: Bad! [said in a passionate tone of voice and body language expressing dissatisfaction]
> PT Peter: Bad?
> Tim: Because Johan does not like bamsemums.
> Johan: No, it is disgusting! [They continue to negotiate. After a while, PT Peter interrupts.]
> PT Peter: Remember when we talked about respecting various opinions? Respect that some like other candies than you. You should show concern for that and consider that they should get bamsemums since they like them. [PT leaves while they still negotiate.]

Although they agreed, Glen and Tim were dissatisfied with the outcome. PT Peter listened to the students' struggles and showed empathic facial expressions. He was supportive and educative when he engaged, reminding them to respect various opinions. Teachers are essential in supporting students in respecting peers' perspectives, and PT Peter did not suggest specific solutions but instead left it to the students to solve their disagreement. He allowed his students to negotiate without losing their independence. The three students were engaged in fair decision-making and defended their stance partly through quantitative thinking. They were kindly reminded of the rules of engagement rather than being presented with solutions.

These two examples illustrate interesting dilemmas. If people constrain their interests in altruistic ways at the expense of the majority, what happens if the majority are unfair or wrong? And if the majority operates in inequitable, biased, or unjust ways, and the students yield to the pressure from the majority? In these situations, it is essential that students become comfortable standing their ground and pushing back toward the majority rather than just "going with the flow." Students can acknowledge the responsibility to ensure decisions favor all students. Students who can make convincing arguments can potentially flip the majority, and by doing so, they are empowered to speak up for the minority. In democracies like the classroom, students do not always have to agree; however, they need to coexist. In Glen's case, we observed that he withdrew a bit from the group discussions after this situation, possibly due to not being heard about his preferences for candy.

### *Bringing in New Perspectives and Missed Opportunities*

Participants in dialogues can sometimes bring in new perspectives that others, for various reasons, miss out on. In Group 1, Chris expressed, "Candy King is better because they show how many grams of sugar your candy consists of, if it is vegan, and that it does not include [unclear]." He seemed to think aloud, an important quality of inquiry-based dialogues. By reflecting upon a specific producer of sweets, he brought in a new perspective. Chris shared his everyday knowledge with peers and highlighted a preference for producers who show transparency about ingredients.

Being a critical citizen can include shedding light on a company's role in increasing the visibility of candies' ingredients, which is an example of an essential feature in reflective knowing and awareness of the formatting powers of mathematics (Skovsmose, 2023). By highlighting that some candies are vegan, Chris showed awareness of vegan food trends. Maybe he was engaged in animal welfare or environmental impact or he was simply lactose intolerant and was therefore concerned with this issue. Either way, it shows that students can develop critical modeling competencies and insight into the relationship between mathematics and reality.

Chris's utterance can invite his peers to discuss these perspectives. However, Levi and Emma replied, "Okay, now we start calculating." They seemingly neglected his argument and did not inquire about this perspective. Although the situation facilitated Chris's right to speak, his peers did not show the associated responsibility to offer him the opportunity to be heard. Accepting Chris's invitation to inquire about this perspective may be considered off topic by Levi and Emma. Instead, they wished to focus on calculating, and perhaps they considered this activity more relevant in the mathematics classroom than discussing candy

producers' transparency. However, deciding which assumptions are appropriate is integral to modeling, and students can be encouraged to negotiate about assumptions. To have the will and courage to pose inquiries, as Chris made, and the curiosity to accept such lines of inquiries is vital for students' democratic practices in the classroom. One could question whether mathematics education provides spaces to discuss issues like these.

### *Which Tools and Approaches Do We Take?*

The PTs provided multiple tools for the students to use during the activity. Deciding which tool to use can be challenging for students in modeling activities and cause disagreement. In Group 4, they initially decided to use Centicubes, but Johan changed his mind and said, "This sucks! I will spend a much longer time using these. Are you in on skipping these?" Johan suggested cutting Centicubes and presenting argumentation. Tim and Glen agreed, and they stopped using Centicubes. Thus, they jointly settled on the choice of tools. Later, when calculating the grams of candies, they started to discuss the use of another tool, a calculator:

> Glen: I want to use a calculator.
> Johan: No, that's not allowed.
> Glen: Yes, yes, yes . . . I want to use the calculator. [PT Anna arrives]
> Glen: Could we use a calculator to calculate?
> PT Anna: No, you cannot.
> Johan: Okay, twenty-eight. Seventeen plus eleven is twenty-eight.

Glen wanted to use a calculator when adding, but Johan reminded him about the classroom rules. When PT Anna arrived, she answered Glen's question by saying no. She provided no reason why the students could not use a calculator. There might be good reasons why she did not want them to use the calculator; for instance, she wanted them to practice addition or multiplication skills. However, following Wright's (2020) ideas about a socially just pedagogy, she could explain the reasons behind this choice (relational authority) instead of using her position of power to impose the rules (positional authority). She denied Glen using the calculator, leaving no room for negotiation spaces. The students did not object to the PT's negative response and immediately started calculating. One could imagine that the students were empowered to negotiate back. They could argue that calculating was time-consuming, and by using the calculator, they could shift from focusing on operational competencies toward more use of mathematics in the modeling activity instead.

### *What Are Fair Criteria for Sharing?*

When the PTs asked the students to "share the candies in your group as fairly as possible," this formulation left room for negotiation. Compared to typical mathematical textbook tasks that have one obvious answer, for instance, asking students to divide 300 grams by four people, where the answer is 75 grams, this task has multiple outcomes and solutions. Anna and Peter did not initially explain what "fairly" meant and let students discuss in groups freely. Later, in plenary discussions, they asked the students how they interpreted fair sharing. When the students negotiated in groups, they discussed criteria like uneven sharing, order of choosing candies, and practical considerations. Examples of uneven sharing were students saying, "We can give candies to another group" or "You could give me more candies." Maybe they just aimed for an easy solution. However, one could assume that they gave away candies to strive for maximum fairness within the group. Alternatively, one could interpret it as a form of kindness toward less privileged groups, where they exercised a form of hypothetical altruism toward groups consisting of four members instead of three.

The order of choosing candies was rooted in students' own experiences with sharing candies with siblings. From the plenary discussion, one student said: "We get to choose what kind of candies first, then we get what we want. Then they [siblings] get more candies than us, just that they don't get to choose." He suggested that uneven sharing was okay and acknowledged that choosing first matters most, probably because he preferred some candies. Thus, the value of the candy meant more than the amount of candy. Taking a strictly mathematical approach by sharing by number (either in grams or pieces) thus seemed to be discarded by this student. Instead, he used his informal knowledge and preferences to decide on fair criteria. Thus, the students practiced a form of reflective knowing, deciding that using numbers is not purposeful in this situation.

Another example of practical considerations the students made was "If my big brother and I chose different candies each weekend, and our mum should stay in the kitchen and weigh[. . .], it would take hours." The student who said this used her everyday knowledge and provided arguments that weighing the candy is time-consuming and found it unreasonable to expect her mother to weigh all the candies and divide them into equal parts. She concluded the siblings themselves should choose. Her experience was used to dismiss the mathematical formatting of weighing the candy as unpractical. Finding the optimal solutions through mathematical modeling can forward answers to real-world problems. In this case, she reasoned that maximum fairness (in this case, equal weight) would worsen the situation for others (her mother).

## Concluding Comments and Implications

When identifying how democratic practices occur during the modeling activity "The Candy Bag of Dreams," we found relevant features in both PT and student practices. The PTs chose an optimizing modeling activity, where students should find the best possible solution among a set of options. This task was easily accessible to the students, aligning with what Greefrath et al. (2022) argued. It involved maximum fairness, as described by Eley (2013) and Suh et al. (2021) and provided spaces for students' negotiation (Barbosa, 2007; Frejd & Bergsten, 2018). When students discussed maximum fairness when sharing candies, they included criteria like uneven sharing, order of choosing, practical considerations, and individual perspectives versus the collective. Qualities of such negotiations are relevant to students' critical and social modeling competencies (Cevikbas et al., 2022).

During the modeling activity, the PTs supported students' negotiations, respected various opinions, and facilitated their reflection on fair criteria. They invited the students to think mathematically in equitable discussions, where their inquiries are pursued and valued (Hansen, 2021). Modeling can provide spaces for democratic practices like students' negotiation (Barbosa, 2007). The PTs probed various questions about what fair criteria for sharing candies could be. This allowed the students not only to focus on mathematical abstractions like grams or numbers but also to move their attention toward nonformal techniques and asked them if they could handle these issues without necessarily using mathematics in line with reflective knowing, as described by Skovsmose (2023). The modeling activity allowed the PTs to invite the students to inquire about a situation where the PTs had set the scene with real candies (Skovsmose, 2023). It guided the students without explicitly telling them what to do, for instance, by enabling them to negotiate without losing their independence.

### *Empowerment and Mathemacy*

Our findings and discussions indicate that PTs engaged the students in a mathematics classroom *for* and *as* democracy (Jansen et al., 2021). When students exercised their quantitative thinking when finding the optimal number of candies or sharing them, they were learning competencies relevant for democracy. When presenting arguments about taste and weight when negotiating about a particular candy, they practiced their argumentation skills to defend a stance; it was about learning relevant practices for democracy (Jansen et al., 2021). When students decided which candies to choose or how to share, they were engaged in collective and fair decision-making and experienced the mathematics classroom as a democracy (Jansen et al., 2021). In these situations, they became empowered to speak up for themselves, and such empowerment is essential for democratic practices to take place (Skovmose, 2023).

Empowerment can also include moral principles and concerns for others' well-being, and when choosing candies, we observed examples where students constrain their self-interest by being fair to the majority, as described by Mansbridge (1990). However, balancing the majority rule with individual rights can create tensions in classrooms, and students and teachers need to consider how to act in such situations. In particular, they should consider how to engage in discussions if the majority is unfair or wrong. Students use their everyday experiences and mathematical approaches to find ways that make sense when choosing and sharing candies. We also discussed an example where one student expressed interest in and thought aloud about the transparency of specific producers of candies; he showed a beginning awareness of, and competency to interpret and understand his social reality. Such mathemacy is essential to question, challenge, and critically examine how mathematics is used in society (Skovmose, 2023). However, the students' thoughts were not pursued further during the group work. For teachers, it can be relevant to question what we emphasize as important in the mathematics classroom; is it just the learning of operations, such as calculating, or do we also provide space for student reflections on the role of mathematics in society?

### *Dialogues and Discussions*

During the modeling activity, the PTs actively facilitated student-led communication, which can lead to more democratic talk practices (Hansen, 2021). Students showed various qualities as described in inquiry-based dialogue by Alrø and Johnsen-Høines (2012) and the IC model by Alrø and Skovsmose (2002). When discussing their candy bag, students attempted to get in contact with peers and identify and locate peers' opinions. They let everyone have a say by asking which candy they wanted and trying to maintain equality by voting. When negotiating which tools to use, they advocated for their interests, presented arguments, challenged, and agreed before continuing.

However, acquiring the competencies to have inquiry-based dialogues is not straightforward. In one situation, the student Chris introduced a new perspective to his peers, who were seemingly unwilling to investigate this. Students can experience being not listened to in the classroom by teachers or peers (Tyson et al., 2022). There can be various reasons for students overlooking what peers are saying, and in this situation, perhaps they did not consider this topic to be part of mathematics or just wanted to finish the task. However, as highlighted by Alrø and Johnsen-Høines (2012), willingness to inquire about new perspectives is crucial in inquiry-based dialogues, as is negotiating assumptions and variables when modeling (Frejd & Bergsten, 2018). Therefore, students need to be supported to develop their listening skills and to inquire about perspectives suggested by peers.

### *Rights and Responsibilities*

The PTs organized the modeling activity in groups and contributed to engaging the students to exercise their rights and responsibilities, as described by Jansen et al. (2021). Students exercised their right to speak, claim mistakes, and revise their thinking when they discussed choosing and sharing candy, which was in line with what was highlighted by Prasad and Kalinec-Craig (2021). They practiced their responsibilities, like asking peers to share their thoughts when choosing which candy to include. During students' fierce negotiation about candies, the PTs balanced their governing role in the classroom while supporting students' rights as learners (Jansen et al., 2021). Rather than the PTs suggesting possible solutions to students, they reminded them to respect each other's opinions and left it to them to find a solution. The governing role also came into play when students wished to use a specific tool. Here, the PT used positional authority and classroom rules instead of exercising relational authority and explaining the reasoning behind pedagogical choices in line with a socially just pedagogy (Wright, 2020).

Implications of this research suggest that modeling activities can be a fruitful approach where students can exercise democratic practices. We support Barbosa (2007) and Frejd and Bergsten (2018) in that negotiation should be explicit during modeling activities, and we add that democratic practices should be emphasized. Further investigation could focus on how negotiations could move forward critical and social aspects of modeling competencies and how democratic practices can occur. Also, the context in this research was candy. However, students could benefit from more substantive contexts (e.g., sustainability and equity). Therefore, in

future studies, the context could be a part of the modeling activity and part of democratic practices.

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